# Transformations of the Gaussian Free Field in Arbitrary Dimensions

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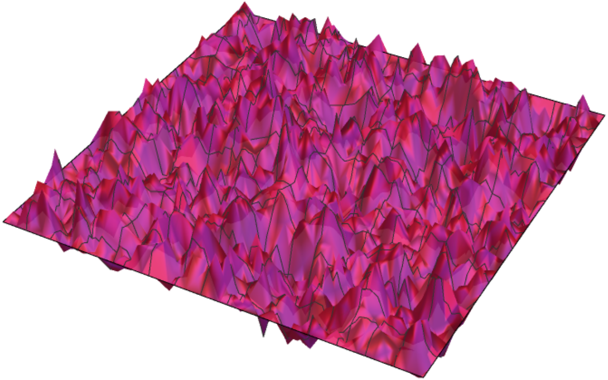
# The Gaussian Free Field: Pretty Picture



**Informally**, the Gaussian Free Field (GFF) in $\mathbb{R}^d$ is a random surface with a certain pointwise distribution. In $d = 1$, this is the usual Brownian motion on $\mathbb{R}$.

# The Gaussian Free Field: Definitions

- Let $(H, (\cdot,\cdot)_{H^1})$ to be the completion of $C_0^\infty(\mathbb{R}^d)$ under

$$(f,g)_{H^p} = \int_{\mathbb{R}^d} (I-\Delta)^p f \cdot g \mathrm{d}x = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1+|\xi|^2)^p \hat{f}(\xi) \cdot \overline{\hat{g}}(\xi) \mathrm{d}\xi.$$

- Take $\alpha_n \sim^{\text{i.i.d.}} \mathcal{N}(0,1)$ standard Gaussian variables and $\{h_n\}$ orthonormal basis.
- Define the series

$$h = \sum_n \alpha_n h_n.$$

**Theorem**

$$\mathbb{E}[h(x)h(y)] \sim \begin{cases} \mathcal{O}(\log|x-y|) & d=2, \\ \mathcal{O}(|x-y|^{2-d}) & d \geq 3 \end{cases}$$, *hence, h is ill-defined pointwise.*

Despite this, we can still interpret $h$ in a **"distributional sense"**.

# Ball Average, $d \geq 3$

We can look at how $h$ behaves when integrated over certain sets. Fix a radius $t > 0$ and position $x \in \mathbb{R}^d$. Let $h_{x,t}$ be the average of $h$ over $B(x, t)$.

**Theorem**

$$\mathbb{E}[h_{x,t}^2] = C_d t^{-d} \int_0^\infty \frac{1}{1+\lambda^2} \frac{1}{\lambda} J_{d/2}^2(t\lambda) \mathrm{d}\lambda \sim \mathcal{O}_x(t^{2-d})$$

As we shrink our ball, the average of our field over said ball grows polynomially, as we expect. In short, this is a "good averaging" (we don't pick up more growth than the natural covariance).

# Images of Ball Averages

Ball averages are **convenient** - radially symmetric. We expect to get similar behaviour when averaging over other sets.

**Theorem (Hu, Miller, Peres)**

*In $d = 2$, if $\psi$ a conformal map, then the difference in the average of $h$ over $B(x, t)$ and the image of $B(x, t)$ under $\psi$, appropriately rescaled, converges to zero as $t \to 0^+$.*

The case for $d \geq 3$ is not known in general, but we have established:

**Theorem**

*If $\psi$ a linear diffeomorphism of $\mathbb{R}^d$, then the ball average and "transformed" ball average under $\psi$ at $x$ are incomparable unless $\psi$ a multiple of a rotation matrix.*

In practice, we need to look at properties of Fourier transforms of certain measures.

# Future Directions

**Conjecture**

The ball and transformed ball averages are incomparable unless $\psi$ locally (first-order) a multiple of a rotation matrix everywhere, i.e.

$$J_\psi(x) \cdot J_\psi^\dagger(x) = c(x) I_d.$$

*This would straightforwardly extend the result in $d = 2$.*

**Conjecture**

The same result holds for the "spherical average", taken as the surface integral over $\partial B(x, t)$ normalized by the surface area of the $t$-radius ball.

*The importance of this second result is that, probabilistically, it is far more convenient to work with the spherical rather than ball average. Analytically, the situation is far more subtle.*