The Discrete Variational Bicomplex

MATH470 Final Report

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Introduction

A *conservation law* of a differential equation $F[u] := F(x; u_x; u_{xx} \dots) = 0$ is a function $\phi[u]$ that is identically constant on solutions on solutions to F, or equivalently

$$\mathsf{D}_x \phi[u] \bigg|_{F[u]=0} = 0, \tag{1}$$

where D_x the total derivative in x. The study of conservation laws is ubiquitous in the study of partial differential equations (PDEs). Conservation laws are useful in the construction of stable discretizations of PDEs [WBN17] [WN18] and the study of key geometric structures underlying many systems, just to name a few applications.

Theorem 1 ([Olv86]). Let ϕ be a conservation law of the system of differential equations $F[u] = 0$. Then, there exists a tuple $Q = (Q_1[u], \dots, Q_\ell[u])$ such that

$$\mathrm{Div}(\phi) = Q \cdot F. \tag{2}$$

Q is called the *multiplier* or *characteristic* of the conservation law ϕ.

Hence, to find conservation laws (or more specifically, divergences of conservation laws), it suffices to seek multipliers Q.

Theorem 2. Let

$$\mathsf{E} = \sum_{\alpha, I} (-\mathsf{D}_x)^I \frac{\partial}{\partial u_I^\alpha} \tag{3}$$

be the *Euler operator*, where α iterates over the dependent variables and the multiindex I iterates over the independent variables. Then,

$$\mathsf{E}(F[u]) = 0 \iff \exists P[u] \text{ s.t. } F = \mathrm{Div} P. \tag{4}$$

In other words, the kernel of the Euler operator is characterized precisely by divergence expressions.

Combining these two theorems gives rise to a systematic method for finding conservation laws. For a differential equation $F = 0$, allowing Q to be some arbitrary "multiplier" (for which we must in general specify a priori how many derivatives of u it depends upon), we aim to solve the differential equation

$$\mathsf{E}(Q \cdot F) = 0. \tag{5}$$

Upon solving for such a Q, $Q \cdot F$ gives the divergence form of a particular conservation law of the system $F = 0$ (solving for the conservation law itself is its own issue, see for instance [Wan10]).

Clearly, the Euler operator is quite important to the study of conservation laws, and there are many ways that it arises naturally, for instance in the computation of extrema for variational problems. Another, more intrinsic manner that it arises is as the composition of two operators in a particular algebraic structure called the *variational bicomplex*. The bicomplex is a, aptly named, double chain complex of differential forms defined over the jet bundle of a smooth manifold. It provides an abstract, geometric framework to study differential equations, and in particular, variational principles. Exactness of this chain complex leads to the aforementioned characterization of the Euler operator, as well as other important results such as a necessary and sufficient condition for a differential equation being the Euler-Lagrange equations for a given functional. See [And9s] for the standard reference.

In this paper, we explore a construction of an analogous chain complex based on an algebra of difference functions defined over a discrete space, X, which we call our "lattice". These difference functions are ultimately meant to represent finite difference approximations of differential equations, but we avoid making this assumption throughout for the sake of generality. That being said, the intention with this construction is to explore connections between the conservation laws of differential equations and their associated difference equation approximations. Seeing as the variational bicomplex is the natural algebraic structure to study these in the smooth world, it only makes sense to attempt to study these in a discrete analog of the variational bicomplex.

In section 1, we construct the discrete variational bicomplex from the ground up, synthesizing ideas from [HM04], [Zha10], and [And9s] to develop the most appropriate structure. We discuss the differences and similarities between what is possible to imitate from the smooth world, and what we may only approximate, carefully. We finally discuss how a discrete analog of the Euler operator arises naturally from the complex, as well as develop a novel generalization in the case of arbitrary difference operators on the complex.

In section 2, we discuss exactness of the complex. Many of these theorems come for free by comparison with the exactness of the de Rham complex, but those involving operations on the lattice are not so clear. In particular, we prove in theorem 9 exactness of the *horizontal complex*, arguably the least transparent aspect of the construction, but also vital for the characterization of the Euler operator. To the best knowledge of the author, this proof is novel. The theorem was first claimed in [HM04] without proof, then argued against in [Zha10] for the proposed homotopy operators not being appropriate. We establish the theorem here.

Finally, in section 3, we discuss what is missing from the theory we've developed so far. In particular, we'd like to make a connection between the discrete and smooth variational bicomplexes. The ultimate goal would be to be able to define projection maps from the smooth to the discrete and related interpolation maps in the opposite direction that commute with the operations on the respective complexes, leading to a notion of consistency. An intermediary goal is to discuss some notion of convergence of the discrete to the smooth, an idea that was briefly discussed in [Hyd01] in the specific case of the discrete versus smooth Euler operators. We also discuss other future directions, such as establishing exactness of the inner rows of the complex, a result which has been discussed but never claimed and especially never proven, and potential applications of our results to understanding conservative discretizations of differential equations.

1 Construction of the Discrete Variational Bicomplex

In this section, we construct our underlying algebra of functions on the lattice, and discuss the relationships between our construction and the standard analog in the smooth theory [And9s]. We begin by formalizing what we mean when we speak of "difference functions" as an analog to differential equations. We then demonstrate how we construct a chain complex on the dependent (vertical, U) and independent (horizontal, X) variables by defining particular modules over rings of functions on the respective spaces and taking their subsequent exterior powers. Then, we construct the entire bicomplex of interest by taking the direct product of these two complexes, and adjoin the variational complex through the consideration of certain cohomological spaces. Finally, we show how this construction naturally gives rise to a discrete analog of the Euler operator discussed in the introduction.

1.1 Functions on a Lattice

Fix positive integers p and q denoting the dimension of our independent, dependent variables respectively. Let $X := \mathbb{Z}^p$, with coordinates $\mathbf{n} := (n_1, \ldots, n_p)$ and fix a real-valued vector space $U \cong \mathbb{R}^q$, with coordinates $(u_\ell \in \mathbb{R} : \ell = 1, \ldots, q)$. Denote the *shift operator* on X by

$$S_i^k : X \longrightarrow X, \quad \mathbf{n} \mapsto \mathbf{n} + k \cdot e_i, \quad i = 1, \ldots, p, \; k \in \mathbb{Z} \tag{6}$$

and $e_i \in X$ the vector of all zeroes except the i-th coordinate equal to 1. More concisely, define for any $K = (k_1, \ldots, k_p) \in \mathbb{Z}^p$ the shift operator

$$S^K := S_1^{k_1} \circ \cdots \circ S_p^{k_p} : X \longrightarrow X, \quad \mathbf{n} \mapsto \mathbf{n} + K. \tag{7}$$

Define the space of real-valued functions on X by

$$\mathcal{A}(X) := \{f : X \longrightarrow \mathbb{R}\}, \tag{8}$$

and extend the shift operators to $\mathcal{A}(X)$ by precomposition, namely

$$S^K : \mathcal{A}(X) \longrightarrow \mathcal{A}(X), \quad f \mapsto S^K f, \quad (S^K f)(\mathbf{n}) := f(S^K \mathbf{n}). \tag{9}$$

Further, we define the "length" of an element $\mathbf{n} = (n_1, \ldots, n_p) \in X$ by

$$|\mathbf{n}| \coloneqq |n_1| + \cdots + |n_p|. \tag{10}$$

Define, for any positive integer k,

$$X_k \coloneqq \{\mathbf{n} \in X : |\mathbf{n}| \leq k\}, \tag{11}$$

which should be viewed as an analog of a closed ball centered at the origin in $\mathbb{R}^p$.

1.2 Discrete Jet Space

In the smooth formulation, differential equations are viewed as functions on jets of smooth functions of a fibered manifold $\pi : E \to M$, where E, M are $p+q$, p dimensional smooth manifolds respectively. For instance, the equation

$$u_t(x,t) = u_{xx}(x,t), \quad x \in \mathbb{R}, t > 0 \tag{12}$$

can be viewed as the function

$$F[x, t; u_x, u_t; u_{xx}, u_{xt}, u_{tt}] = u_t(x,t) - u_{xx}(x,t) \tag{13}$$

defined on $J^\infty(E)$ where $\pi : E \to M$, $E = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, $M = \mathbb{R} \times \mathbb{R}^+$. We denote general functions on the jet space

$$F[u] \coloneqq F[x_1, \ldots, x_n; \partial_{x_i} u; \partial^2_{x_j} u; \ldots]. \tag{14}$$

In our construction of the discrete analog, we'll consider the trivial bundle $\pi : \mathbb{R}^{p+q} \to \mathbb{R}^p$, as generalizing to arbitrary manifolds is not clear as of yet.

We consider the analogous "discrete fibered space"

$$\pi : X \times U \to X, \quad (\mathbf{n}, \mathbf{u}) \mapsto \mathbf{n}. \tag{15}$$

For any positive integer $k \geq 0$, define the space $U^k \coloneqq \{(u_\ell) : \ell \in X_k, u_\ell \in U\}$, and the *$k$-discrete jet spaces* $J^k \coloneqq X \times U^k$, with trivial projections onto the second coordinate

$$\pi_k : J^k \to X. \tag{16}$$

This induces projections for $1 \leq j \leq k$

$$\pi^k_j : J^k \to J^j \tag{17}$$

given by truncation of the first coordinates outside of U^j. Let

$$\mathcal{A}(U^k) \coloneqq C^\infty(U^k, \mathbb{R}) \tag{18}$$

be the space of smooth real-valued functions on U^k for every $k \geq 1$.

Then, define for every positive integer $k \geq 1$,

$$\mathcal{A}(J^k) := \mathcal{A}(X) \otimes \mathcal{A}(U^k). \tag{19}$$

There are then natural projections $\mathcal{A}(J^k) \to \mathcal{A}(J^j)$ for any $j \leq k$ induced by those discussed above. We then finally define our space of functions on the (discrete) jet space by

$$\mathcal{A}(J) := \operatorname{limind}_{k\to\infty} \mathcal{A}(J^k). \tag{20}$$

We remark that this algebra of functions is the same as the "finite order" algebra of functions Zharinov considers in his paper [Zha10], which he denotes $\mathcal{A}_{\text{fin}}(J)$. The characterizing quality of this algebra is the notion of a "global order". Namely, for any function $f(u,k) \in \mathcal{A}(J)$, $f(u,x)$ lives in $\mathcal{A}(J^k)$ for some sufficiently large k, and thus f vanishes outside of $X \times U^k$, *for every input* x.

We could have instead taken a limit earlier in our construction and defined instead $\mathcal{A}(U) = \operatorname{limind}_{k\to\infty} \mathcal{A}(U^k)$, then defined our full function space as $\mathcal{A}(X) \otimes \mathcal{A}(U)$. This algebra does not have the same global order property, and while for each input x a function has finite order, there is certainly no guarantee of a global order. This difference may seem subtle, but as shown in [Zha10], this latter choice leads to a completely trivial variational calculus on this space of functions since there is essentially no relationship between the independent and dependent variables and thus the complex can be "split" completely. Thus, we will for the remainder consider the algebra $\mathcal{A}(J)$, which as we will show admits a nontrivial variational component.

We summarize the discrete, smooth analogs as follows in table 1.

Smooth	**Discrete**
$\pi : E \to M$	$\pi : X \times U \to X$
$\pi_k : J^k(E) \to M$	$\pi_k : J^k \to X$
$\pi^k_j : J^k(E) \to J^j(E)$	$\pi^k_j : J^k \to J^j$

Table 1: Summary of smooth, discrete analogous structures.

1.3 The Exterior Algebra

We have no differential structure in the discrete case, and so the construction of a discrete variational theory in the same language of the smooth world isn't obvious. However, with the correspondence between vector fields and derivations on a manifold in mind, we attempt to replicate the construction of an exterior algebra by considering a module over the algebra of functions defined on our discrete space X, and defining our exterior algebra on top of this space, and its algebraic dual.

As in the smooth case, we can separately consider the exterior algebra over just the discrete space X (analogous to the de Rham complex), and that over the space of dependent variables U. From there, we can derive the full bicomplex by tensoring these spaces together appropriately. We largely follow the construction of [Zha10].

1.4 The Vertical Complex

We consider first the exterior algebra over U. Let

$$\mathcal{D}(U) := \mathsf{Der}(\mathcal{A}(U)) \tag{21}$$

be the $\mathcal{A}(U)$-module of derivations of $\mathcal{A}(U)$, with standard basis given by $\{\partial_{u^\alpha_\ell} : \ell \in X, \alpha = 1, \ldots, q\}$. Let $\mathcal{D}^*(U)$ be the dual module with dual basis given by $\mathrm{d}u^\beta_k : k \in X, \beta = 1, \ldots, \beta$ where $\mathrm{d}u^\beta_k, \partial_{u^\alpha_\ell} = \delta^\ell_k \delta^\alpha_\beta$. For $k \geq 1$ let

$$\Omega^k(U) := \textstyle\bigwedge^k_{\mathcal{A}(U)} \mathcal{D}^*(U), \tag{22}$$

the kth exterior power of the module $\mathcal{D}^*(U)$. A typical element $\omega \in \Omega^k(U)$ is then of the form

$$\omega = \sum_{\substack{(\ell_1,\ldots,\ell_k)\in X^k \\ (\alpha_1,\ldots,\alpha_k)\in\{1,\ldots,q\}^k}} \omega^{\alpha_1,\ldots,\alpha_k}_{\ell_1,\ldots,\ell_k} \mathrm{d}u^{\alpha_1}_{\ell_1} \wedge \cdots \wedge \mathrm{d}u^{\alpha_k}_{\ell_k}, \quad \omega^{\alpha_1,\ldots,\alpha_k}_{\ell_1,\ldots,\ell_k} \in \mathcal{A}(U). \tag{23}$$

We denote the entire exterior algebra by

$$\Omega(U) := \bigoplus_{k\geq 1} \Omega^k(U). \tag{24}$$

The *vertical differential* is then defined as the standard differential on the (now infinite dimensional) space with coordinates u^α_ℓ. To be precise, given $\omega \in \Omega^k(U)$, the vertical differential $\mathrm{d}_V : \Omega^k(U) \to \Omega^{k+1}(U)$ is given

$$\mathrm{d}_V \omega = \sum_{\alpha, I} \frac{\partial}{\partial u^\alpha_I} \omega \wedge \mathrm{d}u^\alpha_I, \tag{25}$$

where the $\frac{\partial}{\partial u^\alpha_I}$ should be viewed as acting coefficient-wise on ω. Because each of the coefficient functions on ω have at most finite dependence on the variables of the form u^α_I, there are no issues with taking this formally infinite sum.

Its easy to see that $\mathrm{d}_V^2 = 0$, so $\{\Omega^k(U), \mathrm{d}_V\}$ defines a complex. Indeed, $\Omega^k(U)$ is really just the de Rham complex in variables u^α_I, namely an infinite dimensional space.

In the U variables, despite lacking real differentiable structure, we had a clear analog with the space of derivations. In the X space, however, no such "natural" construction exists. Instead, we aim to imitate the construction through a somewhat artificial yet representative differentiation counterpart.

We need to define, for $\mu = 1, \dots, p$, operators $\Delta_\mu : X \to X$, imitating differentiation in the x coordinate. In previous literature, the only considered operators have been the forward difference given by $\Delta_\mu = S^{e_\mu} - \mathrm{id}$. However, here we allow general difference operators of the form

$$\Delta_\mu = \sum_{I \in \mathcal{I}(\mu)} c_I^\mu S^I, \tag{26}$$

where $\mathcal{I}(\mu)$ a finite index set of p-tuples $I \in X$ and $c_I^\mu \in \mathbb{R}$. The only other condition we impose on the operators is of mutual commutation, namely

$$\Delta_\mu \circ \Delta_\nu = \Delta_\nu \circ \Delta_\mu, \quad \forall 1 \leq \mu, \nu \leq p, \tag{27}$$

which is crucial for ensuring the complex we build atop these operators obeys the relation $\mathrm{d}_H^2 = 0$, as we will see to follow.

Define then the free $\mathcal{A}(X)$-module spanned by $\{\Delta_\mu : \mu = 1, \dots, p\}$ by

$$\mathcal{D}(X) := \left\{ \sum_{\mu=1}^{p} f_\mu \Delta_\mu : f_\mu \in \mathcal{A}(X) \right\} \subset \mathsf{End}(\mathcal{A}(X)). \tag{28}$$

We then consider the dual module spanned by $\mathrm{d}x_\nu$, where $\mathrm{d}x_\nu(\Delta_\mu) = \delta_\nu^\mu$,

$$\mathcal{D}^*(X) = \left\{ \sum_{\nu=1}^{p} f_\nu \mathrm{d}x_\nu : f_\nu \in \mathcal{A}(X) \right\}. \tag{29}$$

For $0 \leq j \leq p$, define

$$\Omega^j(X) := \textstyle\bigwedge^j_{\mathcal{A}(X)} \mathcal{D}^*(X), \tag{30}$$

and denote the entire exterior algebra

$$\Omega(X) = \bigoplus_{0 \leq j \leq p} \Omega^j(X). \tag{31}$$

To define operations on the bicomplex, we first extend shift operators to $\Omega(X)$ by acting coefficient-wise, namely if $\omega = \sum_I \omega_I \mathrm{d}x^I \in \Omega(X)$ where $\mathrm{d}x^I := \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_j}$, then

$$S^K \omega_I := \sum_I (S^K \omega_I) \mathrm{d}x^I. \tag{32}$$

Define too for $\mu = 1, \ldots, p$,

$$\varepsilon^{\mu} : \Omega^{j}(X) \to \Omega^{j+1}(X), \quad \sum_{I} \omega_I \mathrm{d}x^I \in \Omega(X) \mapsto \mathrm{d}x^{\mu} \wedge \omega. \tag{33}$$

With this, we can define the *exterior horizontal derivative* by summing the composition of these operators over μ,

$$\mathrm{d}_H : \Omega^{j}(X) \to \Omega^{j+1}(X), \quad \mathrm{d}_H = \sum_{\mu=1}^{p} \varepsilon^{\mu} \circ \Delta_{\mu}. \tag{34}$$

In particular, we have the following:

Theorem 3. d_H is a linear map that squares to zero.

Proof. The linearity follows from the linearity of the wedge and shift operators. To see that $\mathrm{d}_H^2 = 0$, we compute directly. Supposing $\omega \in \Omega^r(X)$ of the form $\omega = \sum_I \omega_I \mathrm{d}x^I$, then

$$\mathrm{d}_H \omega = \sum_{\mu} \sum_{I} \Delta_{\mu}(\omega_I) \mathrm{d}x^I \wedge \mathrm{d}x^{\mu}, \tag{35}$$

so

$$\mathrm{d}_H^2 \omega = \sum_{\nu \neq \mu} \sum_{I} \Delta_{\nu}(\Delta_{\mu}(\omega_I)) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}. \tag{36}$$

We have $\Delta_\nu \circ \Delta_\mu = \Delta_\mu \circ \Delta_\nu$ by construction and that $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, so each term in this summation has a corresponding negative canceling perfectly and hence $\mathrm{d}_H^2 = 0$. □

Hence, we have the *discrete difference complex* $\{\Omega^j(X), \mathrm{d}_H\}$. Graphically,

$$0 \longrightarrow X \longrightarrow \Omega^0(X) \xrightarrow{\mathrm{d}_H} \Omega^1(X) \xrightarrow{\mathrm{d}_H} \cdots \xrightarrow{\mathrm{d}_H} \Omega^{p-1}(X) \xrightarrow{\mathrm{d}_H} \Omega^p(X) \xrightarrow{\mathrm{d}_H} 0, \tag{37}$$

where the map $0 \to X$ is the trivial inclusion and the map $X \to \Omega^0(X)$ maps constants to constant functions.

1.6 The Bicomplex

With the constructions of the vertical and horizontal complexes $\Omega(U)$ and $\Omega(X)$, we may define the entire *discrete variational bicomplex*

$$\Omega = \Omega(J) := \bigoplus_{\substack{k \geq 1 \\ 0 \leq j \leq p}} \Omega^{k,j}(J) \tag{38}$$

where the $\mathcal{A}(J)$-modules are given

$$\Omega^{k,j} = \Omega^{k,j}(J) := \Omega^k(U) \wedge_{\mathbb{R}} \Omega^j(X). \tag{39}$$

We note that we do limit the underlying function space to $\mathcal{A}(J)$ as defined previously. Operations on the algebras $\Omega(X), \Omega(U)$ extend naturally to Ω. For operations on $\Omega(U)$, operations simply act invariantly on components from $\Omega(X)$ (in particular, one can view functions on $\mathcal{A}(J)$, in the context of $\Omega(U)$, as functions on U with the X variables acting as parameters). For operations on $\Omega(X)$, we have

$$S^I f = f \circ S^I, \quad S^I \mathrm{d}u_J^\alpha = \mathrm{d}u_{I+J}^\alpha, \tag{40}$$

for any $f \in \mathcal{A}(J)$ and $\mathrm{d}u_I^\alpha$ vertical one-form.

Theorem 4. The necessary connecting relation,

$$\mathrm{d}_V \circ \mathrm{d}_H + \mathrm{d}_H \circ \mathrm{d}_V = 0, \tag{41}$$

holds. Moreover, $\mathrm{d}_V^2 = \mathrm{d}_H^2 = 0$ still holds; in particular, $\{\Omega^{k,j}; \mathrm{d}_V, \mathrm{d}_H\}$ is a vertically unbounded, horizontally bounded bicomplex.

Proof. The fact that $\mathrm{d}_V^2 = 0$ is clear since differentiation in the vertical components treats X variables as parameters. $\mathrm{d}_H^2 = 0$ requires additional proof, since the shift operators also affect the vertical forms. Supposing $\omega \in \Omega^{r,s}$ of the form $\omega = \sum_{\alpha,J,I} \omega_I^{\alpha,J} \mathrm{d}u_J^\alpha \wedge \mathrm{d}x^I$, then

$$\mathrm{d}_H \omega = \sum_\mu \sum_{\alpha,J,I} \Delta_\mu(\omega_I^{\alpha,J} \mathrm{d}u_J^\alpha) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^\mu, \tag{42}$$

so

$$\mathrm{d}_H^2 \omega = \sum_{\nu \neq \mu} \sum_{\alpha,J,I} \Delta_\nu(\Delta_\mu(\omega_I^{\alpha,J} \mathrm{d}u_J^\alpha)) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{43}$$

By the same logic as in theorem 3, this entire expression is identically zero.

We show the proof for the connecting relation in the case $p = q = 1$ and $\omega \in \Omega^{0,0}$ for notational simplicity and to demonstrate the crucial steps. By direct computation, one finds

$$\begin{aligned}
(\mathrm{d}_H \circ \mathrm{d}_V)(\omega) &= \mathrm{d}_H(\sum_i \frac{\partial \omega}{\partial u_i} \mathrm{d}u_i) && (44)\\
&= \sum_i \sum_{I \in \mathcal{I}(1)} S^I[\frac{\partial \omega}{\partial u_i} \mathrm{d}u_i] \wedge \mathrm{d}x^1 && (45)\\
&= \sum_i \sum_{I \in \mathcal{I}(1)} S^I[\frac{\partial \omega}{\partial u_i}] \, \mathrm{d}u_{i+I} \wedge \mathrm{d}x^1, && (46)
\end{aligned}$$

and similarly

$$(\mathrm{d}_V \circ \mathrm{d}_H)(\omega) = \sum_i \sum_{I \in \mathcal{I}} \frac{\partial}{\partial u_i}(S^I \omega)\mathrm{d}x^1 \wedge \mathrm{d}u_i. \tag{47}$$

We can compare coefficients corresponding to $\mathrm{d}u_i \wedge \mathrm{d}x^1, \mathrm{d}x^1 \wedge \mathrm{d}u_i$ in each of these expressions respectively. In the first, coefficients are $S^I \frac{\partial \omega}{\partial u_{i-I}}$, and in the second $\frac{\partial}{\partial u_i}(S^I \omega)$, which we find to be equal by the important relation

$$S^I \circ \frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_{i+I}} \circ S^I. \tag{48}$$

This, combined with the fact that $\mathrm{d}u_{i+I} \wedge \mathrm{d}x^1 = -\mathrm{d}x^1 \wedge \mathrm{d}u_{i+I}$ completes the proof. □

1.7 The Euler-Lagrange Complex

In order to speak of a discrete variational calculus, we need to extend our complex slightly. We consider for every nonnegative integer k the cohomology spaces

$$\mathcal{F}^k := \Omega^{k,p} / \mathrm{d}_H \Omega^{k,p-1}. \tag{49}$$

Namely, two (k, p)-forms are equal in $\mathcal{F}^k$ if and only if they differ by an exact form. Such forms are typically called *functional forms* as they serve the role of functionals in the theory of the calculus of variations. Denote $\pi^k : \Omega^{k,p} \to \mathcal{F}^k$ the natural quotient map. Define the *variational derivative* by

$$\delta : \mathcal{F}^k \to \mathcal{F}^{k+1}, \quad \delta = \pi^{k+1} \circ \mathrm{d}_V \circ (\pi^k)^{-1}. \tag{50}$$

We immediately have the following theorem.

Theorem 5. δ as defined above is well-defined. In particular, $\delta^2 = 0$.

Proof. We need to show that d_V is zero on total horizontal forms, i.e. if $\omega = \mathrm{d}_H \eta$, then $\mathrm{d}_V \omega = 0$, which is clear from direct computation. □

Thus, $\{\mathcal{F}^k, \delta\}$ defines a complex. Combined with the bicomplex constructed in the previous sections gives the entire *discrete variational bicomplex*, fig. 1. In particular, the outer row, of especial importance, is typically called the *Euler-Lagrange complex*, fig. 2.

$$
\begin{tikzcd}
 & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 & & \Omega^{2,0} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\mathrm{d}_H"] & \Omega^{2,1} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\mathrm{d}_H"] & \cdots \arrow[r, "\mathrm{d}_H"] & \Omega^{2,p-1} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\mathrm{d}_H"] & \Omega^{2,p} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\pi^2"] & \mathcal{F}^2 \arrow[u, "\delta"] \\
 & & \Omega^{1,0} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\mathrm{d}_H"] & \Omega^{1,1} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\mathrm{d}_H"] & \cdots \arrow[r, "\mathrm{d}_H"] & \Omega^{1,p-1} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\mathrm{d}_H"] & \Omega^{1,p} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\pi^1"] & \mathcal{F}^1 \arrow[u, "\delta"] \\
0 \arrow[r, hook] & J \arrow[r, hook] & \Omega^{0,0} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\mathrm{d}_H"] & \Omega^{0,1} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\mathrm{d}_H"] & \cdots \arrow[r, "\mathrm{d}_H"] & \Omega^{0,p-1} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\mathrm{d}_H"] & \Omega^{0,p} \arrow[u, "\mathrm{d}_V"] \arrow[r, "\pi^0"] & \mathcal{F}^0 \arrow[u, "\delta"] \\
0 \arrow[r, hook] & X \arrow[r, hook] & \Omega^0(X) \arrow[u, hook] \arrow[r, "\mathrm{d}_H"] & \Omega^1(X) \arrow[u, hook] \arrow[r, "\mathrm{d}_H"] & \cdots \arrow[r, "\mathrm{d}_H"] & \Omega^{p-1}(X) \arrow[u, hook] \arrow[r, "\mathrm{d}_H"] & \Omega^p(X) \arrow[u, hook] &
\end{tikzcd}
$$

Figure 1: The Discrete Variational Bicomplex

$$
J \longrightarrow \Omega^{0,0} \xrightarrow{\mathrm{d}_H} \Omega^{0,1} \xrightarrow{\mathrm{d}_H} \cdots \xrightarrow{\mathrm{d}_H} \Omega^{0,p-1} \xrightarrow{\mathrm{d}_H} \Omega^{0,p} \xrightarrow{\pi^0} \mathcal{F}^0 \xrightarrow{\delta} \mathcal{F}^1 \xrightarrow{\delta} \cdots
$$

Figure 2: The Discrete Euler-Lagrange Complex

1.7.1 The Euler Operator

Of particular interest is the map $\delta : \mathcal{F}^0 \to \mathcal{F}^1$. Denoting a horizontal top form by $\mathrm{d}x^T$, let $\omega = f\mathrm{d}x^T \in \Omega^{0,p}$ where $f \in \mathcal{A}(J)$. Computing the vertical differential for such an expression, we find

$$\mathrm{d}_V\omega = \sum_{\substack{\alpha=1,\ldots,q\\ I=(i_1,\ldots,i_p)}} \frac{\partial f}{\partial u_I^\alpha}\mathrm{d}x^T \wedge \mathrm{d}u_I^\alpha \tag{51}$$

$$= \sum_{\alpha,I} \frac{\partial f}{\partial u_I^\alpha} S^I(\mathrm{d}x^T \wedge \mathrm{d}u_0^\alpha). \tag{52}$$

Let's assume for now that we're working with our difference operators all being forward differences, namely $\Delta_\mu = S^{e_\mu} - \mathrm{id}$ for $\mu = 1, \ldots, p$. Then, notice that we have

$$\mathrm{d}_H((S^{-I}\frac{\partial f}{\partial u_I^\alpha})\mathrm{d}u_0^\alpha) = \sum_\mu \Delta_\mu(S^{-I}\frac{\partial f}{\partial u_I^\alpha})\mathrm{d}u_0^\alpha) \wedge \mathrm{d}_H^\mu \tag{53}$$

$$= \sum_\mu \left[(S^{e_\mu - I}\frac{\partial f}{\partial u_I^\alpha})S^{e_\mu}(\mathrm{d}u_0^\alpha \wedge \mathrm{d}_H^\mu) - (S^{-I}\frac{\partial f}{\partial u_I^\alpha})\mathrm{d}u_0^\alpha \wedge \mathrm{d}_H^\mu\right]. \tag{54}$$

Passing to the quotient space $\mathcal{F}^1$, then the left-hand side of this expression is equivalent to zero, and so

$$\sum_\mu \left[(S^{e_\mu - I}\frac{\partial f}{\partial u_I^\alpha})S^{e_\mu}(\mathrm{d}u_0^\alpha \wedge \mathrm{d}_H^\mu)\right] = \sum_\mu \left[(S^{-I}\frac{\partial f}{\partial u_I^\alpha})\mathrm{d}u_0^\alpha \wedge \mathrm{d}_H^\mu\right], \quad \text{modulo } \mathrm{d}_H. \tag{55}$$

Then, upon changing index of summation, the expression in eq. (52) may be rewritten in terms of those on the left-hand side of eq. (55), yielding

$$\delta\omega = \sum_{\alpha,I}(S^{-I}\frac{\partial f}{\partial u_I^\alpha})\mathrm{d}u_0^\alpha \wedge \mathrm{d}_H^\mu \tag{56}$$

$$= \sum_\alpha \sum_I \left[S^{-I}\frac{\partial f}{\partial u_I^\alpha}\right]\mathrm{d}u_0^\alpha \wedge \mathrm{d}_H^\mu \tag{57}$$

$$= \sum_\alpha \mathsf{E}^\alpha(f)\mathrm{d}u_0^\alpha \wedge \mathrm{d}_H^\mu, \tag{58}$$

where we define the *Euler operator* for each $\alpha = 1, \ldots, q$ by

$$\mathsf{E}^\alpha : \mathcal{A}(J) \to \mathcal{A}(J), \quad f \mapsto \sum_I S^{-I}\frac{\partial f}{\partial u_I^\alpha}. \tag{59}$$

In particular, by passing to the quotient $\mathcal{F}^1$, we were able to effectively "mod out" all of the dependence on vertical forms except for $\mathrm{d}u_0^\alpha$ for each $\alpha = 1, \ldots, q$. In other words, the variational derivative of a form ω is determined by a single value for each vertical dimension, given by E^α. One should note the similarities with the traditional "smooth" Euler operator, under the identification of the total derivative with S^{-I}.

In this section thus far, we've restricted our attention to forward difference operators, as is standard in the literature. In doing so, we've remarked that by modding out by total difference d_H, we can iteratively simplify the variational derivative of a function to depend on a single vertical form. Heuristically, one can think about this quotienting as a manner of associating shifts. Suppose for simplicity that $p = q = 1$. If $\mathrm{d}_H = 0$, then we are in a sense saying that $S = \mathrm{id}$. This identification is what made it possible to rewrite $\mathrm{d}_V\omega$ with the only vertical form being $\mathrm{d}u_0^\alpha$, since then

$$0 = \Delta \mathrm{d}u_0^\alpha = (S - \mathrm{id})\,\mathrm{d}u_0 \implies \mathrm{d}u_0 = \mathrm{d}u_1 \quad \text{modulo } \mathrm{d}_H. \tag{60}$$

We generalize this type of association to more general Δ. We remain working with $p = q = 1$ for simplicity. Suppose

$$\Delta = \sum_{i=1}^{M} c_i S^{s_i}, \tag{61}$$

where c_i's are nonzero constants, and the s_i's are distinct shifts arranged such that $s_1 < s_2 < \cdots < s_M$. Working modulo d_H and thus modulo Δ, we find

$$S^{s_M} = \sum_{i=1}^{M-1} -\frac{c_i}{c_M} S^{s_i}. \tag{62}$$

Without loss of generality, we may assume that $s_1 = 0$ and thus that $s_i \geq 0$ for every $i = 1, \ldots, M$ (namely, all of the shifts are positive), by composing both sides with S^{s_i} if this isn't the case. Define $m \coloneqq s_M - s_{M-1}$ as the gap between the largest and second largest shifts. Then, composing both sides with S^m,

$$\begin{aligned}
S^{s_M+m} &= \sum_{i=1}^{M-1} -\frac{c_i}{c_M} S^{s_i+m} && (63)\\
&= \sum_{i=1}^{M-2} -\frac{c_i}{c_M} S^{s_i+m} - \frac{c_{M-1}}{c_M} S^{s_M} && (64)\\
&= \sum_{i=1}^{M-2} \left[-\frac{c_i}{c_M} S^{s_i+m} \right] - \frac{c_{M-1}}{c_M} \left(\sum_{i=1}^{M-1} -\frac{c_i}{c_M} S^{s_i} \right) && (65)\\
&= \sum_{\hat{s}=0}^{s_M+m-1} C_{\hat{s}} \cdot S^{\hat{s}}, && (66)
\end{aligned}$$

where $C_{\hat{s}}$ are constants depending solely on the constants c_i and the shift $\hat{s}$, whose precise form can be elucidated on inspection of the preceding formula. In short, working modulo d_H allows us to rewrite the shift S^{s_M+m} linearly in terms of lower shifts. Arguing inductively, then, any such operator with shift higher than $s_M + m$ can be then reduced in terms of shifts ranging between 0 and $s_M - 1$. For general S^k, let us denote by $C_{\hat{s}}(k)$ for $\hat{s} = 0, \ldots, s_M + m - 1$ the sequence of constants such that

$$S^k = \sum_{\hat{s}=0}^{s_M+m-1} C_{\hat{s}}(k) S^{\hat{s}}. \tag{67}$$

Again, such constants can be found inductively. Since $S^{k+1} = S \circ S^k$, one finds the recursive relationship

$$S^{k+1} = \sum_{\hat{s}=0}^{s_M+m-1} \left[C_{\hat{s}-1}(k) - C_{s_M+m-1}(k) \cdot \frac{c_{\hat{s}}}{c_M} \right] S^{\hat{s}}, \tag{68}$$

where the indexed constants are taking to be zero if no such index exists. Thus, we find

$$C_{\hat{s}}(k+1) = C_{\hat{s}-1}(k) - C_{s_M+m-1}(k) \frac{c_{\hat{s}}}{c_M}, \quad \forall \hat{s} = 0, \ldots, s_M + m - 1. \tag{69}$$

One can then solve this for arbitrary $k \geq s_M + m$.

Hence, repeating the argument from eq. (55), we find that for a form $\omega = f \, \mathrm{d}x \in \Omega^{0,1}$,

$$\delta\omega = \sum_{\hat{s}=0}^{s_M+m-1} S^{\hat{s}} \left[\sum_i C_{\hat{s}}(i) \cdot S^{-i}\left(\frac{\partial f}{\partial u^i}\right) \right] \mathrm{d}u_{\hat{s}} \wedge \mathrm{d}x \tag{70}$$

where the bracketed term can hence be viewed as a generalized Euler operator, determined solely by the particular Δ chosen. One readily verifies that the Euler operator derived previously arises from this formula upon substitution of the appropriate constants.

As in the smooth world, exactness of the Euler-Lagrange complex gives rise to the precise characterization of the kernel of the Euler operator. We establish this in the case of Δ being a forward difference operator, though the general case is still open.

2 Exactness of the Bicomplex

We discuss now the exactness of the bicomplex. While this has been discussed in several areas [Zha10], [HM04], [PH23], we remark on differences in the exposition presented here. In particular, the proof of the exactness of the horizontal complex has, to the best of our knowledge, not been laid out explicitly before. While [Zha10] presents a proof, it is over a far larger underlying algebra of functions than we have used here which is impractical in application (indeed, it leads to a completely trivial variational calculus). In particular, the functions in their paper do not have "global finite order", and the proof of the exactness of the horizontal complex relies crucially on this fact. [Hyd01], using an algebra of functions more clearly associated with ours, also state this theorem but without proof.

As in the rest of the literature, we present only proofs of the exactness of the outer rows of the complex, and the variational complex. The exactness of the entire bicomplex is still open. In general, proofs of the interior rows of the smooth bicomplex require the notion of "contact forms" on the underlying fibered manifold, a notion that does not have a clear analog in the discrete case. In addition, throughout the remainder of the paper we take our difference operators to simply be the forward difference operators,

$$\Delta_\mu = S^{e_\mu} - \mathrm{id}, \tag{71}$$

as the exactness of the complex has not yet been established in the general case.

Theorem 6 (Exactness of the Vertical Complexes)**.** For every $k = 1, \ldots, p$, the unbounded vertical complex

$$0 \longrightarrow \Omega^{0,k} \xrightarrow{\mathrm{d}_V} \Omega^{1,k} \xrightarrow{\mathrm{d}_V} \Omega^{2,k} \xrightarrow{\mathrm{d}_V} \cdots \tag{72}$$

is exact.

Proof. We follow the proof of [HM04], though it is standard. Define the (formally infinite) "vector field"

$$\mathbf{v} = \sum_{\alpha, I} u_I^\alpha \frac{\partial}{\partial u_I^\alpha} \tag{73}$$

and the homotopy map

$$\mathcal{H}_v : \Omega^{i+1,k} \longrightarrow \Omega^{i,k}, \quad \omega \mapsto \int_0^1 \mathbf{v} \lrcorner \omega[\lambda u] \frac{\mathrm{d}\lambda}{\lambda}, \tag{74}$$

where $\omega[\lambda u]$ denotes ω with each occurrence of a variable in u_J^β replaced with $\lambda \cdot u_J^\beta$. We claim

$$\mathrm{d}_V \mathcal{H}_v + \mathcal{H}_v \mathrm{d}_V = \mathrm{id}. \tag{75}$$

One will notice that this is the same homotopy map commonly used in the proof of the exactness of the de Rham complex; indeed, the vertical complex is nothing more than the de Rham complex in variables u_I^α, with the lattice variables acting as parameters. The infinite dimensionality of the underlying space may seem concerning, but by construction each function in our underlying algebra $\mathcal{A}(J)$ relies on only finitely many variables in u_J^α and thus all summations are necessarily finite. □

Theorem 7 (Exactness of the Variational Complex)**.** The *variational complex*

$$0 \longrightarrow \mathcal{F}^0 \xrightarrow{\delta} \mathcal{F}^1 \xrightarrow{\delta} \mathcal{F}^2 \xrightarrow{\delta} \cdots \tag{76}$$

is exact.

Proof. Since the spaces $\{\mathcal{F}^s\}$ are just quotients of the spaces $\{\Omega^{s,p}\}$, we can actually employ the homotopy operator from the proof of the exactness of the vertical complex. □

Theorem 8 (Exactness of the Difference Complex)**.** The difference complex

$$0 \longrightarrow \Omega^0(X) \xrightarrow{\mathrm{d}_H} \Omega^1(X) \xrightarrow{\mathrm{d}_H} \cdots \xrightarrow{\mathrm{d}_H} \Omega^p(X) \xrightarrow{\mathrm{d}_H} 0 \tag{77}$$

is exact.

Proof. This is the first theorem in which we must depart from the setting of the de Rham complex on the variables u, as our operator d_H cannot be realized as a genuine differential as in the previous two cases. See [HM04] for the proof. We will need to refer to the relevant homotopy operator later, which we will denote as $\mathcal{H}_d$. □

Theorem 9 (Exactness of the Horizontal Complex)**.** The horizontal complex

$$0 \longrightarrow \Omega^{0,0} \xrightarrow{\mathrm{d}_H} \Omega^{0,1} \xrightarrow{\mathrm{d}_H} \cdots \xrightarrow{\mathrm{d}_H} \Omega^{0,p-1} \xrightarrow{\mathrm{d}_H} \Omega^{0,p} \xrightarrow{\mathrm{d}_H} 0 \tag{78}$$

is exact.

Before we present the proof, we need a few lemmas and technical tools.

For two positive multiindices $I = (i_1, \ldots, i_p), J = (j_1, \ldots, j_p)$, we write $I \subseteq J$ if $i_k \geq j_k$ for every $k = 1, \ldots, p$. We also define the "size" of a multiindex by

$$\sharp I = i_1 + \cdots + i_p \tag{79}$$

and define the binomial coefficient of I and J by

$$\binom{I}{J} := \binom{i_1}{j_1} \cdots \binom{i_p}{j_p}. \tag{80}$$

With this we define, for each $\alpha = 1, \ldots, q$ and positive multiindex J, the *higher Euler operators*

$$\mathsf{E}_\alpha^J := \sum_{I \supset J} \binom{I}{J} S^{-I} \frac{\partial}{\partial u_I^\alpha}. \tag{81}$$

Note that while each higher Euler operator an infinite sum, in practice only finitely many terms will be nonzero.

We define too the notion of a *horizontal interior product*, an analog of the natural interior product in the vertical complex in the horizontal component. For $\omega \in \Omega^{0,r}$, where $r = 1, \ldots, p$, we define

$$\mathsf{I}_u := \sum_{I=(i_1,\ldots,i_p)} \sum_{\alpha=1}^{q} \sum_{k=1}^{p} \frac{i_k + 1}{p - r + \sharp I + 1} (S - \mathrm{id})^I (u_{\mathbf{0}}^\alpha \mathsf{E}_\alpha^{I+e_k} (\partial_{x^k} \lrcorner \omega)), \tag{82}$$

where $(S - \mathrm{id})^I = (S - \mathrm{id})^{i_1} \circ \cdots \circ (S - \mathrm{id})^{i_p}$.

One should remark at this point the clear similarities in the definitions presented thus far and those used in the proof of the same theorem in the smooth case, see for instance [Olv86], Chapter 5.4. Indeed, the main difference is the exchange of smooth differential operators (D in Olver) for our discrete difference operators ($S - \mathrm{id}$).

Lemma 1. Let $J \geq K$ be *positive* multiindices in X. Then,

$$\sum_{I \supset K}^{J} (-1)^{\sharp I - \sharp K} \binom{I}{K} \binom{J}{I} = \delta_K^J = \delta_{k_1}^{j_1} \cdots \delta_{k_p}^{j_p}, \tag{83}$$

where δ_a^b the Kronecker delta function, equal to 1 if $a = b$, 0 otherwise.

Proof. This follows from direct computation. □

Lemma 2. Let $f \in \mathcal{A}(J)$. Then,

$$\sum_{\alpha, I} u_I^\alpha \frac{\partial f}{\partial u_I^\alpha} = \sum_{\alpha=1}^{q} \sum_{I} (S - \mathrm{id})^I u_0^\alpha \mathsf{E}_\alpha^I (f). \tag{84}$$

Proof. We prove in the case $q = 1$ since clearly this expression extends by linearity for arbitrary q. Fix some multiindex J. The only terms in the right-hand side of eq. (84) containing $\frac{\partial}{\partial u_J}$ will then be those for which $I \leq J$, namely

$$\sigma_J := \sum_{I \subseteq J} (S - \mathrm{id})^I u_0 \binom{J}{I} S^{-J} \frac{\partial}{\partial u_J} \tag{85}$$

$$= \sum_{I \subseteq J} \left(\prod_{k_\bullet} (-1)^{i_\bullet - k_\bullet} \binom{i_\bullet}{k_\bullet} S_\bullet^{k_\bullet} \right) u_0 \left[\binom{J}{I} S^{-J} \frac{\partial}{\partial u_J} \right], \tag{86}$$

where the expansion in the second line comes from applying binomial expansion to the $(S - \mathrm{id})^I$ term. Fix some other multiindex K. The only terms in σ_J involving such a K will be those for which $I \supseteq K$, namely

$$s_K := \sum_{I=K}^{J} (-1)^{\sharp I - \sharp K} \binom{I}{K} \binom{J}{I} S^K u_0 S^{-J} \frac{\partial}{\partial u_J} \tag{87}$$

$$= u_K S^{K-J} \frac{\partial}{\partial u_J} \sum_{I=K}^{J} (-1)^{\sharp I - \sharp K} \binom{I}{K} \binom{J}{I} \tag{88}$$

$$= u_K S^{K-J} \frac{\partial}{\partial u_J} \delta_K^J, \tag{89}$$

where the last equality holds by lemma 1. We have then that

$$\sigma_J = \sum_{K=I}^{J} s_K \tag{90}$$

$$= \sum_{K=I}^{J} u_K S^{K-J} \frac{\partial}{\partial u_J} \delta_K^J \tag{91}$$

$$= u_J \frac{\partial}{\partial u_J}. \tag{92}$$

So, we find that indeed

$$\sum_I (S - \mathrm{id})^I u_0 \sum_{J \supseteq I} \binom{J}{I} S^{-J} \frac{\partial}{\partial u_J} = \sum_J \sigma_J = \sum_J u_J \frac{\partial}{\partial u_J}, \tag{93}$$

proving the lemma as needed. □

Lemma 3. For any $\alpha = 1, \ldots, q$, index $\ell = 1, \ldots, p$ and multiindex I, we have

$$\mathsf{E}_\alpha^I \circ (S_\ell^1 - \mathrm{id}) = \mathsf{E}_\alpha^{I - e_\ell}. \tag{94}$$

Proof. We prove in the case $p = q = 1$, with the general cases following easily since the only effect of the composition in the lemma is on the independent variables, and the shift S_ℓ only has an effect on the ℓ-th component of the independent variables. We have that

$$\mathsf{E}^i(S^1) = \sum_{j \geq i} \binom{j}{i} S^{1-j} \frac{\partial}{\partial u_{j-1}} \tag{95}$$

$$= \sum_{\tilde{j} \geq i-1} \binom{\tilde{j}+1}{i} S^{-\tilde{j}} \frac{\partial}{\partial u_{\tilde{j}}} \tag{96}$$

$$= \sum_{\tilde{j} \geq i-1} \binom{\tilde{j}}{i-1} S^{-\tilde{j}} \frac{\partial}{\partial u_{\tilde{j}}} + \sum_{\tilde{j} \geq i-1} \binom{\tilde{j}}{i} S^{-\tilde{j}} \frac{\partial}{\partial u_{\tilde{j}}} \tag{97}$$

$$= \mathsf{E}^{i-1} + \mathsf{E}^i. \tag{98}$$

Thus,

$$\mathsf{E}^i \circ (S - \mathrm{id}) = \mathsf{E}^{i-1} \circ S - \mathsf{E}^i = \mathsf{E}^{i-1} + \mathsf{E}^i - \mathsf{E}^i = \mathsf{E}^{i-1}. \tag{99}$$

□

The following three lemmas are all standard properties of operations on chain complexes, adapted to our notations. See for instance [Olv86] for the proofs.

Lemma 4. For any r-length multiindex K where $1 \leq r \leq p$, we have

$$\sum_{k=1}^{p} \partial_{x^k} \lrcorner (\mathrm{d}x^k \wedge \mathrm{d}x^K) = (p - r)\mathrm{d}x^K. \tag{100}$$

Lemma 5. For any $\omega \in \Omega^{0,r}$ for $r = 0, \ldots, p$ and any index $k = 1, \ldots, p$,

$$\partial_{x^k} \lrcorner (\mathrm{d}x^k \wedge \omega) = \omega - \mathrm{d}x^k \wedge (\partial x^k \lrcorner \omega). \tag{101}$$

Lemma 6. For any $\omega \in \Omega^{0,r}$ for $r = 0, \ldots, p$ and any indices $k, \ell = 1, \ldots, p$,

$$\partial_{x^k} \lrcorner (\mathrm{d}x^\ell \wedge \omega) = -\mathrm{d}x^\ell \wedge (\partial_{x^k} \lrcorner \omega). \tag{102}$$

Proof of Theorem 9. For the remainder of the proof, we fix some $\omega \in \Omega^{0,r}$ for $1 \leq r \leq p - 1$. We define the *total homotopy operator*

$$\mathcal{H}_h : \Omega^{0,r} \to \Omega^{0,r-1}, \quad \omega \mapsto \int_0^1 \mathsf{I}_u(\omega)[\lambda u_0] \frac{\mathrm{d}\lambda}{\lambda}, \tag{103}$$

where we use the notation $\mathsf{I}_u(\omega)[\lambda u_0]$ to represent first evaluation of I_u on ω, then the result evaluated at λu_0 (namely, each dependent variable u_I is replaced with $\lambda \cdot u_I$, and the independent variables are left as given). We claim that

$$\mathrm{d}_H \mathcal{H}_h(\omega) + \mathcal{H}_h(\mathrm{d}_H \omega) = \omega - \omega_H, \tag{104}$$

where ω_H defined by setting all u_J's in ω to zero; in particular, ω_H naturally lives in $\Omega^r(X)$.

We have that, for any function $F \in \mathcal{A}(J)$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} F[\lambda u] = \frac{1}{\lambda} \sum_{\alpha, I} u_I^\alpha \frac{\partial F}{\partial u_I^\alpha}[\lambda u]. \tag{105}$$

Integrating both sides with respect to λ from 0 to 1 and substituting the expression from lemma 2,

$$F - F_H = \int_0^1 \sum_{\alpha, I} u_I^\alpha \frac{\partial F}{\partial u_I^\alpha}[\lambda u] \frac{\mathrm{d}\lambda}{\lambda} \tag{106}$$

$$= \int_0^1 \left(\sum_\alpha \sum_I (S - \mathrm{id})^I u_0^\alpha \mathsf{E}_\alpha^I(P) \right) [\lambda u] \frac{\mathrm{d}\lambda}{\lambda} \tag{107}$$

Supposing $\omega = \sum_I \omega_I \mathrm{d}x^I$ for $\omega_I \in \mathcal{A}(J)$, we may extend the above to ω by making operations act coefficient-wise, namely we will write

$$\frac{\partial \omega}{\partial u_I^\alpha} = \sum_J \frac{\partial \omega_J}{\partial u_I^\alpha} \mathrm{d}x^I. \tag{108}$$

We would like to show that the bracketed integrand term in eq. (107) is equal to $\mathrm{d}_H \circ \mathsf{I}_u + \mathsf{I}_u \circ \mathrm{d}_H$, evaluated on ω. We inspect first the term $\mathsf{I}_u(\mathrm{d}_H \omega)$:

$$\mathsf{I}_u(\mathrm{d}_H \omega) = \sum_{\alpha, I} \sum_{k,\ell=1}^{p} \frac{i_k + 1}{p - r + \sharp I} (S - \mathrm{id})^I \left\{ u_0^\alpha \mathsf{E}_\alpha^{I + e_k} \left[\partial_{x^k} \lrcorner (S_\ell - \mathrm{id})(\mathrm{d}x^\ell \wedge \omega) \right] \right\} \tag{109}$$

$$= \sum_{\alpha, I} \frac{(S - \mathrm{id})^I}{p - r + \sharp I} u_0^\alpha \underbrace{\sum_{k,\ell=1}^{p} (i_k + 1) \mathsf{E}_\alpha^{I + e_k} \left[\partial_{x^k} \lrcorner (S_\ell - \mathrm{id})(\mathrm{d}x^\ell \wedge \omega) \right]}_{:=\sigma}. \tag{110}$$

We consider the two cases for the double summation $\sigma = \sigma_{k=\ell} + \sigma_{k \neq \ell}$, depending on whether or not $k = \ell$. When $k \neq \ell$, we may apply lemma 6 and lemma 3 and the fact that $\lrcorner$, $S_\ell - \mathrm{id}$ commute and notice that

$$\mathsf{E}_\alpha^{I + e_\ell} [\partial_x^k \lrcorner (S_\ell - \mathrm{id})(\mathrm{d}x^\ell \wedge \omega)] = -\mathsf{E}_\alpha^{I + e_\ell - e_k} [\mathrm{d}x^\ell \wedge \partial x^k \lrcorner \omega]. \tag{111}$$

When $k = \ell$, we can apply lemma 3 again as well as the linearity of the higher Euler operators:

$$\sigma_{k=\ell} = \sum_{k=1}^{p} \mathsf{E}_\alpha^I[\partial_{x^k} \lrcorner (\mathrm{d}x^k \wedge \omega)] + \sum_{k=1}^{p} i_k \mathsf{E}_\alpha^I[\partial_{x^k} \lrcorner (\mathrm{d}x^k \wedge \omega)] \tag{112}$$

$$= \mathsf{E}_\alpha^I \left[\left(\sum_{k=1}^{p} \partial_{x^k} \lrcorner (\mathrm{d}x^k \wedge \omega) \right) + \sum_{k=1}^{p} i_k \left(\partial_{x^k} \lrcorner (\mathrm{d}x^k \wedge \omega) \right) \right] \tag{113}$$

$$= \mathsf{E}_\alpha^I \left[(p - r)\omega + \omega \sum_{k=1}^{p} i_k - \sum_{k=1}^{p} i_k (\mathrm{d}x^k \wedge (\partial x^k \lrcorner \omega)) \right] \tag{114}$$

$$= (p - r + \sharp I)\mathsf{E}_\alpha^I(\omega) - \sum_{k=1}^{p} i_k \mathsf{E}_\alpha^I \left[\mathrm{d}x^k \wedge (\partial_{x^k} \lrcorner \omega) \right] \tag{115}$$

where in the second line we applied lemma 4 to the inner summation on the left and lemma 5 to the right. Combining then $\sigma_{k=\ell} + \sigma_{k\neq\ell}$, we find

$$\sigma = (p - r + \sharp I)\mathsf{E}_\alpha^I(\omega) - \sum_{k=1}^{p} i_k \mathsf{E}_\alpha^I(\mathrm{d}x^k \wedge (\partial_{x^k} \lrcorner \omega)) - \sum_{k,\ell=1}^{p} (i_k + 1)\mathsf{E}_\alpha^{I+e_\ell-e_k}(\mathrm{d}x^k \wedge (\partial_{x^k} \lrcorner \omega)) \tag{116}$$

$$= (p - r + \sharp I)\mathsf{E}_\alpha^I(\omega) - \sum_{k,\ell=1}^{p} (i_k + 1 - \delta_\ell^k)\mathsf{E}_\alpha^{I+e_\ell-e_l}(\mathrm{d}x^k \wedge (\partial_{x^k} \lrcorner \omega)). \tag{117}$$

Hence, returning to our expression for $\mathsf{I}_u(\mathrm{d}_H\omega)$, we find

$$\mathsf{I}_u(\mathrm{d}_H\omega) = \left[\sum_{\alpha,I} (S - \mathrm{id})^I u_0^\alpha \mathsf{E}_\alpha^I(\omega) \right] - \sum_{\alpha,I} \sum_{k,\ell=1}^{p} \frac{i_k + 1 - \delta_\ell^k}{p - r + \sharp I} (S - \mathrm{id})^I u_0^\alpha \mathsf{E}_\alpha^{I+e_\ell-e_k}(\mathrm{d}x^k \wedge (\partial_{x^k} \lrcorner \omega)) \tag{118}$$

$$= \left[\sum_{\alpha,I} (S - \mathrm{id})^I u_0^\alpha \mathsf{E}_\alpha^I(\omega) \right] - \mathrm{d}_H \mathsf{I}_u(\omega), \tag{119}$$

where the identification in the last line follows from the change of index $I \longrightarrow I - e_\ell$. So, we've shown then

$$\mathsf{I}_u(\mathrm{d}_H\omega) + \mathrm{d}_H\mathsf{I}_u(\omega) = \sum_{\alpha,I} (S - \mathrm{id})^I u_0^\alpha \mathsf{E}_\alpha^I(\omega). \tag{120}$$

This is precisely what we set out to prove, so all together, returning to eq. (106) evaluated with ω in place of F, we have

$$\omega - \omega_H = \int_0^1 [\mathsf{I}_u(\mathrm{d}_H\omega) + \mathrm{d}_H\mathsf{I}_u(\omega)]\,[\lambda u] \frac{\mathrm{d}\lambda}{\lambda} \tag{121}$$

$$= \mathcal{H}_h(\mathrm{d}_H\omega) + \mathrm{d}_H\mathcal{H}_h(\omega), \tag{122}$$

proving the claimed property of the operator $\mathcal{H}_h$.

To complete the proof of exactness, we need to find the "inverse image" of ω_H under d_H. But since ω_H lives in $\Omega^r(X)$, we may simply apply the difference homotopy operator $\mathcal{H}_d$ from theorem 8 and, by linearity, complete the proof. Namely, supposing ω is such that $\mathrm{d}_H\omega = 0$, then

$$\omega = \mathrm{d}_H(\mathcal{H}_h(\omega) + \mathcal{H}_d(\omega_H)) = \mathrm{d}_H((\mathcal{H}_h + \mathcal{H}_d \circ \pi_H)(\omega)). \tag{123}$$

□

3 Notions of Convergence, Inner Exactness and Other Further Directions

3.1 Convergence and the Relationship to the Smooth Bicomplex

Much of the original theory of the discrete variational bicomplex was developed by Hydon and Mansfield in [HM04]. Their approach is much more applied than the one we take here, with their underlying complex structure defined in a very formal, symbolic way. While advantageous for their particular purposes, it makes it difficult to make any meaningful association between the discrete and smooth bicomplexes.

Our approach, modeled largely after [Zha10], is more respectful of the original categories of objects underlying the complex structure in the smooth case. For instance, the difference complex $\{\Omega^k(X), \mathrm{d}_H\}$ is defined using the k-th exterior powers of the dual of a module defined over the ring of functions on X. Other than the fact that the underlying space X is discrete, every other object defined here exists in the smooth case; this approach maintains much more algebraic structure, by its very construction. Hydon, Mansfield define the same structure by stating that their underlying difference forms have certain desired properties; we inherit these properties through deliberate definitions.

This being said, while our approach makes the relationship clearer between the smooth and discrete complexes, it certainly does not make it transparent. Namely, we would ultimately like to speak of some sense of convergence of the discrete complex to the smooth. We present here our running concept of what this "sense" could be, specifically in the lattice variables.

Suppose $p = 1$. We can then restrict $\mathcal{A}(X)$ to $\ell^2(\mathbb{R})$ of square-summable real-valued sequences, for convenience. For sequences $x_\bullet, y_\bullet \in \ell^2(\mathbb{R})$, denote the ℓ^2 norm

$$\langle x_\bullet, y_\bullet \rangle = \sum_{i=-\infty}^{\infty} x_i y_i. \tag{124}$$

We immediately remark that by making this restriction, while we have no Leibniz rule, we have an integration by parts-type property

$$\langle x_\bullet, \Delta y_\bullet \rangle = -\langle \Delta x_\bullet, S y_\bullet \rangle, \tag{125}$$

where we consider $\Delta = S - \mathrm{id}$ the forward difference operator. Supposing we fix some projection map $\pi_\varepsilon : L^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ and an interpolation map $\pi_\varepsilon^* : \ell^2(\mathbb{R}) \to L^2(\mathbb{R})$ depending on some small parameter ε, then given a function $F \in L^2$, we would like to talk about the convergence of

$$\|(\pi_\varepsilon^* \circ \Delta \circ \pi_\varepsilon)(F) - \mathsf{d}F\|_2 \tag{126}$$

as $\varepsilon \to 0^+$, where $\|\cdot\|_2$ denotes the L^2 norm. Namely, we would like to characterize the admissible maps π_ε, π_ε^*, and especially Δ, which would naturally have to be defined with respect to one another, that would allow for eq. (126) to converge to zero.

We also, for a full theory, need to discuss the convergence of differential forms, and this is where the advantage of our construction is clearer. In the smooth case, a differential form $\mathsf{d}x$ is a function on the space of vector fields defined on a manifold M, denoted $\mathfrak{X}(M)$, returning a $C^\infty(M)$ real-valued function. Alternatively, one can view $\mathsf{d}x$ as an element of the algebraic dual of space of derivations of $C^\infty(M)$, which is in a one-to-one correspondence with $\mathfrak{X}(M)$. Similarly, a discrete difference form $\mathrm{d}x^\mu$ acts on the space of derivations of $\mathcal{A}(X)$, $\mathcal{D}(X)$, to return a real-valued function on the lattice in $\mathcal{A}(X)$.

Adopting the same notations for the projection, interpolation maps as before, we say that the discrete difference form $\mathrm{d}x^\mu$ converges to a differential form $\mathrm{d}x$ if for every vector field $v \in \mathfrak{X}(M)$,

$$\|\pi_\varepsilon(\langle \mathrm{d}x^\mu, \pi_\varepsilon(v)\rangle) - \langle \mathrm{d}x, v\rangle\|_2 \to 0 \tag{127}$$

as $\varepsilon \to 0^+$. This naturally extends, by $\mathcal{A}(X)$-linearity, to convergence of arbitrary elements of $\mathcal{D}^*(X)$. Being able to characterize precisely when this convergence occurs would certainly be a step forward, however, how to properly define $\pi_\varepsilon, \pi_\varepsilon^*$ is unclear. See section 3.1 for a graphical summary of the maps we've discussed.

3.2 Exactness of the Inner Rows

As we remarked previously in section 2, exactness of the inner rows of the bicomplex, fig. 1,/ have not been established. While for our purposes this is inconsequential as it is the Euler-Lagrange complex, fig. 2, that is of real interest, the exactness of the inner rows of the complex is worth investigating in the name of seeking a complete analog to the smooth theory. While the other proofs of exactness mirrored quite

$$
\begin{tikzcd}
L^2 \arrow[r, "\mathsf{d}"] \arrow[d, "\pi_\varepsilon"] & L^2 \arrow[d, "\pi_\varepsilon"'] \\
\ell^2 \arrow[r, "\Delta"'] \arrow[u, bend left, "\pi_\varepsilon^*"] & \ell^2 \arrow[u, bend right, "\pi_\varepsilon^*"']
\end{tikzcd}
$$

(a) Convergence of difference operators.

$$
\begin{tikzcd}
\mathfrak{X}(M) \arrow[r, "\langle \mathsf{d}x, \cdot \rangle"] \arrow[d, "\pi_\varepsilon"] & C^\infty(M) \arrow[d, "\pi_\varepsilon"'] \\
\mathcal{D}(X) \arrow[r, "\langle \mathsf{d}_H, \cdot \rangle"'] \arrow[u, bend left, "\pi_\varepsilon^*"] & \mathcal{A}(X) \arrow[u, bend right, "\pi_\varepsilon^*"']
\end{tikzcd}
$$

(b) Convergence of difference forms.

Figure 3: Diagrams of relationship between smooth (top) and discrete (bottom) horizontal operations.

closely those of the smooth bicomplex, the tools used in the proof of the inner rows do not have clear associations in the discrete world. Namely, one makes use (see, for instance, [And9s]) of *contact forms*, which are intrinsically defined with respect to a differential structure on the underlying space, which as we've iterated many times, simply does not exist in the discrete world.

How to define the discrete counterpart of a contact form is unclear, as [HM04] has mentioned. Further, there is no obvious alternative approach to proving this said exactness, and so this topic certainly requires further investigation.

3.3 Applications to Finite Difference Discretizations

[HDP06], [Hyd01] and [HM04] demonstrated how to calculate conservation laws of difference equations using the Euler operator, following a methodology identical to that discussed in the introduction, with obvious modifications. In particular, [HM04] demonstrated how one can achieve a convergence result of the discrete Euler operator to the smooth operator under an appropriate continuum limit. In section 1.7.1, we demonstrated how to obtain generalized Euler operators suited to arbitrary difference operators Δ. We would like to be able to make a statement about the convergence of such operators to the genuine Euler operator as well. In particular, we'd like to be able to compare the rates of convergence of the operators. Namely, one should expect an Euler operator based on higher-order difference operators to admit better approximations to genuine conservation laws. As is clear from the complicated formula above, it is far from clear how to proceed in this direction.

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