Symmetries & Differential Equations

Applications of Lie Groups

By: Louis Meunier
Mentor: William Holman-Bissegger

Directed Reading Program
McGill University
04/30/2024

Contents

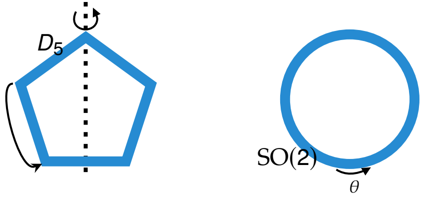
Motivating question: how can we use symmetries to work with differential equations?

Given a solution, can we find a way to continuously map to other solutions?

We will:

- Describe a structure for describing continuous symmetries
- State a requirement for a differential equation to admit a symmetry

Discrete to Continuous Symmetries



Lie group: **differentiable** manifold with **differentiable** operation, inversion.

$\rightsquigarrow$ Provides a structure for **continuous symmetries**

Ex) Planar rotations SO(2), reals $\mathbb{R}$ under addition

Manifold: looks "pretty close" to a subset of $\mathbb{R}^n$: circle, sphere, torus, etc

The Lie Algebra

On G live vector fields; functions that assign vectors to points.

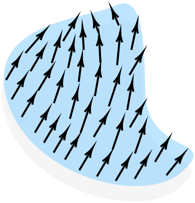
$$\mathbf{v} : M \longrightarrow TM$$

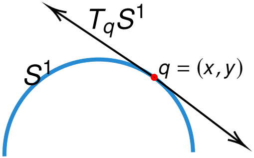
$$\mathbf{v}|_x = \xi^1(x)\partial_{x^1} + \cdots + \xi^m(x)\partial_{x^m}$$

Lie algebra of G : $\mathfrak{g} :=$ {"group operation compatible" vector fields}

Vector fields give rise to **flows**: we denote $\exp(\varepsilon \mathbf{v})x$

$$\mathbf{v}|_x = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \exp(\varepsilon \mathbf{v})x \Big|_{\varepsilon=0} \qquad \exp(\varepsilon \mathbf{v})x = x + \varepsilon\xi(x) + \mathcal{O}(\varepsilon^2)$$





$\rightsquigarrow$ **Linearized**

The Lie Algebra

$$\mathfrak{g} \xrightarrow{\exp} G$$

We now have an **algebraic** means of working with our **group**.
We can replace **non-linear** conditions that arise from the group with **linear** conditions in the algebra.

Rather than thinking of how a group operation behaves, we can think of how the corresponding vector field "flows".

("Because working with a group is cursed" - William)

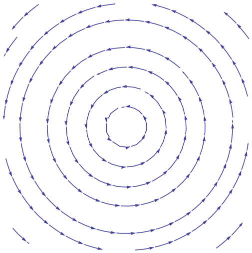
SO(2): Vector Field of a Rotation

$$(x, y) \overset{g_\theta}{\rightsquigarrow} (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$$

$$\mathbf{v}|_{(x,y)} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,(x \cos\varepsilon - y \sin\varepsilon, x \sin\varepsilon + y \cos\varepsilon)\Big|_{\varepsilon=0} = -y\partial_x + x\partial_y$$

Re-deriving the group operation:

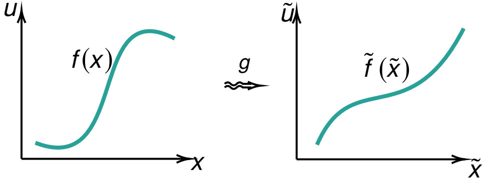
$$\frac{\mathrm{d}x}{\mathrm{d}\varepsilon} = -y, \frac{\mathrm{d}y}{\mathrm{d}\varepsilon} = x \implies g_\varepsilon,$$



Extending to Differential Equations

$\rightsquigarrow$ We can find the effect of a group transformation on a function by "flowing" the function on $\mathbf{v}$; we work on the space

$$(x, u) \in X \times U = \text{ dependent } \times \text{ independent}$$



Transforms **functions**, so transforms **derivatives** of functions - how? Formally, we use "prolongation theory" and "prolonged vector fields"

$$\mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{i=1}^{p} \phi^{x^i} \partial_{u_{x^i}} + \cdots$$

defined over the "jet space" $X \times U^{(n)}$ of $X \times U$.

Infinitesimal Criterion for Invariance

Theorem (Symmetry Condition)

For a differential equation $\Delta(x, u^{(n)}) = 0$ and group of transformations G, if for every infinitesimal generator $\mathbf{v}$ of G,

$$\mathrm{pr}^{(n)}(\mathbf{v}[\Delta(x, u^{(n)})]) = 0 \quad \textit{whenever} \quad \Delta(x, u^{(n)}) = 0,$$

then G is a symmetry group of Δ; G maps solutions to other solutions.

$\rightsquigarrow$ If I flow a differential equation along a vector field, and the result also satisfies the differential equation, then I have a symmetry of the equation.

"Infinitesimal invariance criterion"

An Aside: Noether's Theorem

Theorem (Noether's First Theorem)

If $\mathbf{v}$ *generates a symmetry group of the variational problem* $L[u]$*, then there exists a corresponding* ***conservation law*** *of the Euler-Lagrange equations* $\mathcal{E}(L) = 0$.

$\rightsquigarrow$ Connection between **symmetries** and **conservation laws**

If we allow "generalized symmetries"

$$\mathbf{v} = \sum_{i=1}^{p} \xi_i[u] + \sum_{i=1}^{q} \phi_i[u]$$

this becomes a **one-to-one** correspondance.

The Heat Equation: Characterization of Symmetry Groups

Given a **differential equation** $\Delta = 0$, how do we find all symmetries of the equation? We consider the heat equation

$$\Delta(t, x, u^{(2)}) = u_{xx} - u_t = 0,$$

and an arbitrary vector field

$$\mathbf{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \phi(x, t, u)\partial_u,$$

for which the **infinitesimal invariance criterion** (using prolongation) is

$$\phi^t = \phi^{xx}.$$

We expand this to find a system of PDEs for the coefficients of the vector fields $\mathbf{v}$, and solve to find all independent vector fields.

Heat Equation: Symmetry Groups

Ex)

$$\mathbf{v}_2 = \partial_t \Rightarrow \exp(\varepsilon \mathbf{v})(x,t,u) = (x, t+\varepsilon, u),$$ **Time translation!**

$$\mathbf{v}_4 = x\partial_x + 2t\partial_t \Rightarrow \exp(\varepsilon v)(x,t,u) = (e^{\varepsilon}x, e^{2\varepsilon}t, u),$$ **Space-time scaling!**

Symmetries:

G_i	$\exp(\varepsilon \cdot \mathbf{v}_i)(x,t,u)$	**Physical Interpretation**
1	$(x+\varepsilon, t, u)$	Space Translation
2	$(x, t+\varepsilon, u)$	Time Translation
3	$(x, t, e^{\varepsilon}u)$	Linearity (Constants)
4	$\left(e^{\varepsilon}x, e^{2\varepsilon}t, u\right)$	Space/Time Scaling
5	$\left(x+2\varepsilon t, t, u\cdot e^{-\varepsilon x-\varepsilon^2 t}\right)$	Galilean Boost
6	$\left(\frac{x}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon x}, u\sqrt{1-4\varepsilon t}e^{\frac{-\varepsilon x^2}{1-4\varepsilon t}}\right)$	?
α	$(x, t, u+\varepsilon\alpha(x,t))$	Linearity (Additivity)

Given a solution to $\Delta = 0$, applying G_i to it yields another solution.

Heat Equation: Fundamental Solution

Note that $\Delta(c) = 0$ for any constant c.

Applying G_6 with $\varepsilon = 1$ and then G_2 with $\varepsilon = -\frac{1}{4}$ to $u = c := \sqrt{\frac{1}{\pi}}$ we have

$$c \overset{G_6}{\rightsquigarrow} \frac{c}{\sqrt{1+4t}} \cdot \exp\left(\frac{-x^2}{1+4t}\right) \overset{G_2}{\rightsquigarrow} \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-x^2}{4t}\right\},$$

the **fundamental solution of the heat equation.**

Conclusion

We:

- Formalized continuous symmetries
- Found infinitesimal invariance criterion for symmetry of differential equations
- Applied to the heat equation

$\rightsquigarrow$ **Tools for describing symmetries of differential equations**

Acknowledgements

Thank you to William Holman-Bissegger for guiding me throughout this semester, as well as Bart Syroka, Hazem Hassan, and Shereen Elaidi, for organizing DRP this year.

References

Olver, P. J. (1986). *Applications of lie groups to differential equations*. Springer-Verlag.

Bluman, G. W., et al. (2010). *Applications of symmetry methods to partial differential equations* (S. Antman et al., Eds.). Springer.

Wan, A. T., et al. (2016). Conservative methods for dynamical systems.

Wan, A. T., et al. (2018). The multiplier method to construct conservative finite difference schemes for ordinary and partial differential equations.

Noether, E. (1918). Invariant variation problems. *Transport Theory and Statistical Physics*, (235-257).